VECTOR CALCULUS
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COMPREHENSIVE TEST FOR CHAPTERS 5–8

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1. (a) Evaluate

$$\iint_D 3(x+y)e^{(x-y)}\,dx\,dy$$

over the region bounded by the lines $x+y=1, x+y=-1, x-y=1$ and $x-y=-1$.

(b) Evaluate

$$\int_0^1 \int_x^1 \sin y^2\,dy\,dx.$$

2. A block has a slanted top described by $x+y+z=2$. Its edges are perpendicular to the xy plane, and the bottom of the block is formed by the triangle with vertices $(1,0,0)$, $(0,-1,0)$, and $(0,1,0)$. What is the volume of this block?

3. Find the mass of a wall described by $0 \le y \le -x^2 - 2x + 15$, $-3 \le x \le 3$, having density $\delta(x,y) = 2y+3$.

4. Let $\mathbf{F}(x,y) = (e^x \cos 3y, -3e^x \sin 3y)$.
(a) Find an $f(x,y)$ such that $\nabla f = \mathbf{F}$ for all (x,y).

(b) Evaluate $\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$ for the path $\mathbf{c}(t) = (\cos t, \sin t), 0 \le t \le 2\pi$.

(c) Compute $\nabla \times \mathbf{F}$.
(You should be able to do parts (b) and (c) in your head.)

5. (a) Suppose water flows through a permeable surface that can be described by $x^2+y^2+2z^2=1$, at the velocity $2x\mathbf{i}+3y\mathbf{j}+z\mathbf{k}$. After adjusting the flow of water, it later flows through the same surface at a velocity of $5x\mathbf{i}+y\mathbf{j}+3z\mathbf{k}$. Thinking of this as a flux of water through a surface, is more or less water flowing through the surface after the adjustment? Explain.

(b) A particle moves in a path $\mathbf{c}(t) = (2t, 3t, t)$ in a force field $\mathbf{F} = (2x, 2y, 4z)$. What is the work done in the time interval $0 \le t \le 5$?

6. (a) A contact lens can be described as a cap of a sphere of radius R cut out by a cone of angle $\pi/4$. Find the surface area of the lens. (Hint: Set up the lens in a correct and convenient coordinate system.)

(b) Slice a sphere anywhere with 2 parallel planes that are separated by a fixed distance d. This is shown in Figure 1. Prove that the bands obtained always have the same surface area. What is the area? (Hint: Use spherical coordinates.)

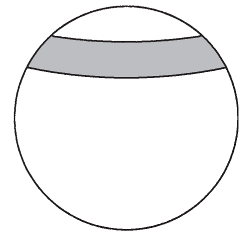

FIGURE 1

7. (a) Find the volume enclosed by the surface

$$x^2 + 9y^2 + z^2 - 2x + 54y - 10z + 106 = 0.$$

(b) Find an expression for the surface area of the surface above.

8. Which of the following vector fields are conservative?

(a) $\mathbf{F}(x, y, z) = (x^2 + 1/x, \ln(y + 1), z)$

(b) $\mathbf{F}(x, y, z) = \left(\dfrac{-3x}{(x^2 + y^2 + z^2)^{3/2}}, \dfrac{-3y}{(x^2 + y^2 + z^2)^{3/2}}, \dfrac{-3z}{(x^2 + y^2 + z^2)^{3/2}} \right)$

(c) $\mathbf{F}(x, y, z) = (3yz, 2xz, 5xy)$

9. Complete the following statement: A vector field $\mathbf{F}$ is conservative if ________.

(i) There is a vector field $\mathbf{G}$ such that $\mathbf{F} = \text{curl } \mathbf{G}$.

(ii) There is a scalar function f such that $\nabla f = \mathbf{F}$.

(iii) $\text{div } \mathbf{F} = 0$.

(a) (i) only (b) (i) and (iii) (c) (ii) only (d) (ii) and (iii).

10. Consider the following argument: given any vector field $\mathbf{F}$,

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}$$

by Stokes' theorem, but

$$\iiint_W \nabla \cdot (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{S=\partial W} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

by Gauss' theorem. Therefore all of these integrals are equal to zero because the divergence of the curl of any vector field is 0. What is wrong with this argument?