VECTOR CALCULUS
J. Marsden and A. Tromba

COMPREHENSIVE TEST FOR CHAPTERS 1–4

1. Let $f(x, y) = x^2 - 2x + y^2 - 1$.
   (a) Find the relative extrema of f.

   (b) Find the absolute extrema of f on the curve $x^2 + y^2 = 1$.

2. Suppose the elevation of a mountain is given by $f(x, y) = e^{2x} \sin y + 3y$. You are standing at the point $\left(0, \frac{\pi}{2}, 1 + \frac{3\pi}{2}\right)$.
   (a) In which direction should you walk in order to descend as quickly as possible?

   (b) You must follow the path $\mathbf{c}(t) = \left(t^2 - 1, \frac{\pi}{2}\right)$. How quickly is your elevation changing when $t = 1$?

3. Show that the path $\mathbf{c}$ is a flow line for the vector field $\mathbf{F}$.
   (a) $\mathbf{F}(x, y) = (y, -x)$; $\mathbf{c}(t) = (\sin t, \cos t)$.

   (b) $\mathbf{F}(x, y) = (2x, -y)$; $\mathbf{c}(t) = (e^{2t}, e^{-t})$.

4. The path $\mathbf{c}(t) = (-\sin t, -\cos t, t\sqrt{15})$, with $1 \le t \le 3$, describes part of a helix. Find the arc length of this path.

5. You want to build a wooden box and you have \$120 to spend on the wood. You want to make the lid of the box out of Douglas fir, which costs \$2 per square inch. The rest of the box will be made of redwood, which costs \$2.50 per square inch. What is the volume of the largest box you can build?

6. (a) The two planes $3x + 2y - z = 7$ and $x - 4y + 2z = 0$ intersect in a line. Find an equation for this line.

   (b) Find the equation of the plane which contains the points $(1, 3, -2)$ and $(0, -2, 1)$ and is perpendicular to the plane $3x - y - 2z = 5$.

7. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $(u, v) \mapsto (uv^2, v^3 - u, v \sin u)$. Let $g\colon \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $(x, y, z) \mapsto (yze^x, y^3 \cos(xz))$.
   (a) Find $\mathbf{D}(f \circ g)(0, 1, 0)$.

   (b) Find $\mathbf{D}(g \circ f)(0, 1)$.

8. Let $\mathbf{a} = (1, 0, \sqrt{3})$ and $\mathbf{b} = (0, 0, 2)$.
   (a) Find the angle between $\mathbf{a}$ and $\mathbf{b}$.

   (b) Find the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$.

9. A plane takes off with constant velocity $3\mathbf{i}+4\mathbf{j}+\mathbf{k}$ until it reaches its cruising altitude at $\left(0, 0, \frac{1}{2}\right)$. It then continues in the same xy-direction with no change in altitude. (Here, distance is measured in miles and time is measured in hours). Assume that the surface of the earth is flat, and is represented by the plane $z = 0$.
   (a) How long did it take to reach cruising altitude?

   (b) What are the xy coordinates of the point of takeoff?

   (c) Three hours after takeoff, the plane passes directly over a pack of wolves. What are the xy coordinates of the pack?

10. Let $f(x, y) = x^2 + x \sin y$. Find the second-order Taylor polynomial for f at $\left(1, \frac{\pi}{2}\right)$.