VECTOR CALCULUS
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SAMPLE TEST FOR CHAPTER 5

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1. True or false. If false, explain why.
   (a) The area between the parabola $y = 4 - x^2$ and the x axis can be calculated using the double integral
   $$\int_0^{4-x^2} \int_{-2}^{2} dx\, dy.$$
   (b) If $g(x, y, z)$ is integrable over a region Q in $\mathbb{R}^3$, then
   $$\iiint_Q g(x, y, z)\, dx\, dy\, dz = \iiint_Q g(x, y, z)\, dy\, dz\, dx.$$
   (c) The following expressions are equal:
   $$\int_0^4 \int_{-2}^4 \int_{y^2}^{5-y^3} (z + y)^{3/2} dz\, dy\, dx = (4) \left(\int_{-2}^4 \int_{y^2}^{5-y^3} (z + y)^{3/2} dz\, dy \right).$$
   (d) For any integrable function $f(x, y)$, the double integral $\iint_D f(x, y)\, dx\, dy$ is the volume of the region bounded by the surfaces $z = f(x, y)$ and $z = 0$ over D.
   (e) The following expressions are equal:
   $$\int_0^2 \int_{-1}^1 (x^2 + 4x + \sqrt{x})(y^3 - y^2 + 4y)\, dy\, dx$$
   $$= \left[\int_0^2 \int_{-1}^1 (x^2 + 4x + \sqrt{x})\, dy\, dx \right] \left[\int_0^2 \int_{-1}^1 (y^3 - y^2 + 4y)\, dy\, dx \right].$$

2. Find the volume under the paraboloid $z = x^2 + y^2$ lying inside the rectangle $(x, y) \in [0, 2] \times [0, 3]$.

3. Change the order of integration and rewrite the triple integral $\int_0^1 \int_{x^2}^1 \int_0^{1-y} dz\, dy\, dx$ in five ways. Compute its value.

4. Find the volume of the region enclosed by the surfaces $z = x^2 + 1$, $z = 0$, $y = 0$ and $y = x^3$, for $-1 \le x \le 2$.

5. Evaluate the double integral $\int_0^{\sqrt{1-\cos(1)}} \int_{\arccos(1-y^2)}^1 y\, dx\, dy$.

6. Evaluate the double integral $\iint_T |x-y|\,dy\,dx$, where T is the triangle with vertices at (2, 0), (0, 0), and (0, 2).

7. (a) Suppose $f : W \to \mathbb{R}$ is continuous and W is an elementary region in $\mathbb{R}^3$. Let $V(W)$ be the volume of W, m the minimum value of f on W, and M the maximum value of f on W. State a theorem that relates the values of m, M, V, and the triple integral of f over W.

   (b) Show that
$$\frac{\pi\sqrt{2}}{16e^2} \le \int_0^1 \int_0^y \int_0^{\pi/4} \frac{\cos(xy)}{e^{y+z}}\,dx\,dz\,dy \le \frac{\pi}{8}.$$

8. Evaluate the integral
$$\int_0^4 \int_0^1 \int_{\sqrt{z}}^2 \exp(x^3)\,dx\,dy\,dz.$$

9. When built, the Great Pyramid of Giza was 280 Egyptian cubits tall, with the square at its base having sides of length 440 cubits each.
   (a) At a height h, a cross section of the pyramid is a square. Write a formula for the area of such a square in terms of h.

   (b) Use Cavalieri's principle to calculate the volume of the pyramid.

   (c) If a coordinate system were placed over the pyramid so that the pyramid's base was in the xy plane, and the tip of the pyramid was at $(0, 0, 280)$, could the pyramid be considered an elementary region in $\mathbb{R}^3$? Why or why not?

10. The region D is shown in Figure 1. The boundary on the left is $y = 4 - x^2$. All of the other boundaries are straight lines.
   (a) Describe D as a union of two elementary regions.

   (b) Use your description from part (a) to find the area of D.

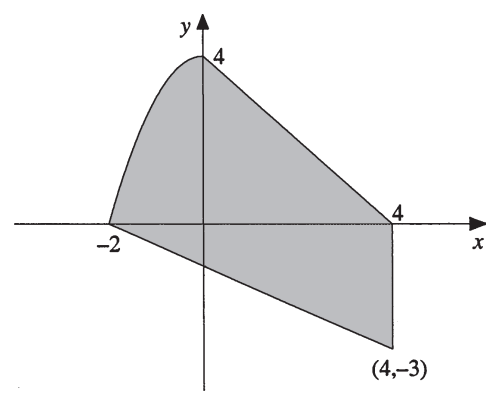

FIGURE 1