PROOF OF L'HÔPITAL'S RULE

In the text, we proved a special case of L'Hôpital's Rule (Theorems 1 and 2 in LT Section 7.7 or ET Section 4.7). This supplement presents the complete proof.

THEOREM 1 Theorem L'hôpital's Rule Assume that $f(x)$ and $g(x)$ are differentiable on an open interval containing a and that

$$f(a) = g(a) = 0$$

Also assume that $g'(x) \neq 0$ for x near but not equal to a. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists. The conclusion also holds if $f(x)$ and $g(x)$ are differentiable for x near (but not equal to) a and

$$\lim_{x \to a} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \pm\infty$$

Furthermore, these limits may be replaced by one-sided limits.

The proof of L'Hôpital's Rule makes use of the following generalization of the Mean Value Theorem known as Cauchy's Mean Value Theorem.

THEOREM 2 Cauchy's Mean Value Theorem Assume that $f(x)$ and $g(x)$ are continuous on the closed interval $[a, b]$ and differentiable on (a, b). Assume further that $g'(x) \neq 0$ on (a, b). Then there exists at least one value c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof First note that $g(x)$ satisfies the hypotheses of the standard Mean Value Theorem on $[a, b]$. Therefore, there exists $r \in (a, b)$ such that

$$g'(r) = \frac{g(b) - g(a)}{b - a}$$

By assumption, $g'(r)$ is not equal to zero. It follows that that $g(b) - g(a)$ is non-zero.

Now, just as in the proof of the standard Mean Value Theorem, we build a new function $h(x)$ to which Rolle's Theorem applies. Set

$$h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g(x)$$

We have not divided by zero since $g(b) - g(a) \neq 0$. Furthermore, $h(x)$ is continuous on $[a, b]$ and differentiable on (a, b) because this is true of both $f(x)$ and $g(x)$. Straightforward calculation yields

$$h(a) = h(b) = \frac{f(a)g(b) - f(b)g(a)}{g(b) - g(a)}$$

Thus, the hypotheses of Rolle's Theorem are satisfied and the conclusion holds, namely, there exists $c \in (a, b)$ such that $h'(c) = 0$. In other words,

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) = 0$$

We may divide by $g'(c)$ (which is nonzero by assumption) and rearrange to obtain the desired equality:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \qquad ■$$

We shall carry out the proof of L'Hôpital's Rule for right-hand limits. The proof for left-hand limits is similar and the result for two-sided limits follows immediately by combining the results for left and right-hand limits. We treat the two cases of L'Hôpital's Rule separately.

■ **CASE 1** $f(a) = g(a) = 0$.

Since $g'(x)$ is non-zero near $x = a$, there is an interval (a, b) such that $g'(x)$ is positive or negative for $x \in (a, b)$, and therefore, $g(x)$ is either strictly increasing or strictly decreasing on (a, b) (Figure 1). However, $g(a) = 0$, so $g(x)$ itself is non-zero for $x \in (a, b)$. Thus for all $x \in (a, b)$, the hypotheses of Cauchy's Mean Value Theorem for the interval $[a, x]$ are satisfied and hence, there exists $c \in (a, x)$ such that

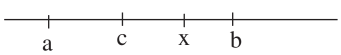

FIGURE 1 Choose b so that $g(x) \neq 0$ for $x \in (a, b)$. Note that $c \to a+$ as $x \to a+$.

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)}$$

Since $f(a) = g(a) = 0$, this reduces to

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

As x tends to a from the right, the value c also tends to a from the right, and the desired conclusion follows:

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f'(c)}{g'(c)} = \lim_{c \to a+} \frac{f'(c)}{g'(c)} = L \qquad ■$$

■ **CASE 2** Infinite Limits.

Now suppose that $f(x)$ and $g(x)$ are differentiable for x near (but not equal to) a and satisfy

$$\lim_{x \to a+} f(x) = \pm\infty \qquad \text{and} \qquad \lim_{x \to a+} g(x) = \pm\infty$$

We assume that

$$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = L$$

By the formal definition of right-hand limits, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$L - \epsilon < \frac{f'(x)}{g'(x)} < L + \epsilon \quad \text{for all} \quad a < x < a + \delta \tag{1}$$

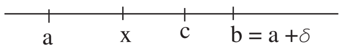

FIGURE 2 We choose b so that (1) holds for all $x \in (a, b)$

Set $b = a + \delta$. Making δ smaller if necessary, we may assume that $f(x)$ and $g(x)$ are differentiable on (a, b), continuous on $[a, b]$ and $g(x) \neq 0$ for $x \in (a, b)$. For every $x \in (a, b)$, the hypotheses of Cauchy's Mean Value Theorem are satisfied on (x, b), so there exists c in (x, b) such that (Figure 2)

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(x)}{g(b) - g(x)} = \frac{\frac{f(x)}{g(x)} - \frac{f(b)}{g(x)}}{1 - \frac{g(b)}{g(x)}}$$

Multiplying by the denominator on the right, we rewrite this as follows

$$\frac{f'(c)}{g'(c)}\left(1 - \frac{g(b)}{g(x)}\right) = \frac{f(x)}{g(x)} - \frac{f(b)}{g(x)}$$

or

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)} - \underbrace{\left(\frac{f(b)}{g(x)} - \frac{f'(c)g(b)}{g'(c)g(x)}\right)}_{\text{call this } r(x)}$$

Denote the indicated quantity on the right-hand by $r(x)$. In other words,

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)} - r(x) \tag{2}$$

Now apply (1) with $x = c$:

$$L - \epsilon < \frac{f'(c)}{g'(c)} < L + \epsilon \tag{3}$$

Together with Eq. (2), we obtain

$$L - \epsilon + r(x) < \frac{f(x)}{g(x)} < L + \epsilon + r(x) \quad \text{for all } x \in (a, b) \tag{4}$$

Keep in mind that b depends on the choice of ϵ. However, for fixed b, we claim that $r(x)$ tends to zero as $x \to a+$. Indeed, the first term $f(b)/g(x)$ tends to zero because $g(x) \to \infty$. Similarly, the second term $f'(c)g(b)/g'(c)g(x)$ tends to zero because $g(x) \to \infty$ and $f'(c)/g'(c)$ remains bounded by (3). Having thus shown that $r(x)$ tends to zero, we may choose $\delta_1 > 0$ such that $|r(x)| < \epsilon$ for all $x \in (a, a + \delta_1)$. We may then apply (4) to conclude that

$$L - 2\epsilon < \frac{f(x)}{g(x)} < L + 2\epsilon \quad \text{for all } x \in (a, a + \delta_1) \tag{5}$$

Since ϵ is arbitrary, this suffices to prove that

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = L$$

as desired. ∎

CASE 3 Limits as $x \to \infty$.

We shall prove L'Hôpital's Rule for limits as $x \to \infty$. The case $x \to -\infty$ is similar. Thus we assume there exists a number b such that $g'(x) \neq 0$ for all $x > b$ and that the following limit exists:

$$L = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

We use the variable $t = x^{-1}$ to convert limits as $x \to \infty$ to one-sided limits as $x \to 0+$. Thus we have

$$\lim_{t\to\infty} \frac{f'(x)}{g'(x)} = \lim_{t\to 0+} \frac{f'(t^{-1})}{g'(t^{-1})} = L$$

By the Chain Rule,

$$\frac{[f(t^{-1})]'}{[g(t^{-1})]'} = \frac{-t^{-2} f'(t^{-1})}{-t^{-2} g'(t^{-1})} = \frac{f'(t^{-1})}{g'(t^{-1})}$$

Therefore

$$\lim_{t\to 0+} \frac{[f(t^{-1})]'}{[g(t^{-1})]'} = \lim_{t\to 0+} \frac{f'(t^{-1})}{g'(t^{-1})} = L$$

Now we may apply L'Hôpital's for one-sided limits as $t \to 0+$ to reach the desired conclusion:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0+} \frac{f(t^{-1})}{g(t^{-1})} = \lim_{t\to 0+} \frac{[f(t^{-1})]'}{[g(t^{-1})]'} = L$$

Remark: In the statement and proof of L'Hôpital's Rule, we have assumed that the limit

$$L = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

is a *finite* limit. This assumption is not necessary: L'Hôpital's Rule remains valid if this limit is infinite (i.e., $L = \pm\infty$). The above proofs may be easily modified to handle this case. For example, suppose that $L = \infty$. In Case 1 ($g(a) = f(a) = 0$), we showed (in the notation of the proof) that

$$\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}$$

Since $c \to a+$ as $x \to a+$, the ratio on the right tends to ∞ as $x \to a+$ and the conclusion follows. Similarly, the proof in Case 2 is valid, but the role of $\epsilon > 0$ is replaced by an arbitrarily large number $M > 0$. Instead of (1), we have, for all M, the existence of $\delta > 0$ such that

$$\frac{f'(x)}{g'(x)} > M \qquad \text{for all } a < x < a + \delta.$$

Instead of (4), we conclude that

$$\frac{f(x)}{g(x)} > M + r(x) \quad \text{for all } x \in (a, b)$$

where $b = a + \delta$. Since $r(x) \to 0$ as $x \to a+$, there exists $\delta_1 > 0$ such that $|r(x)| < \frac{1}{2}M$ for $x \in (a, a + \delta_1)$ and therefore

$$\frac{f(x)}{g(x)} > \frac{1}{2}M \quad \text{for all } x \in (a, a + \delta_1)$$

This proves that $\dfrac{f(x)}{g(x)}$ tends to ∞ as claimed.