CHAPTER 4

PROBABILITY: THE STUDY OF RANDOMNESS

SECTION 4.1

OVERVIEW

A process or phenomenon is called **random** if its outcome is uncertain. Although individual outcomes are uncertain, when the process is repeated a large number of times the underlying distribution for the possible outcomes begins to emerge. For any outcome, its **probability** is the proportion of times, or the relative frequency, with which the outcome would occur in a long series of repetitions of the process. It is important that these repetitions or trials be **independent** for this property to hold.

You can study random behavior by carrying out physical experiments such as coin tossing or rolling of a die, or you can simulate a random phenomenon on the computer. Using the computer is particularly helpful when we want to consider a large number of trials.

GUIDED SOLUTIONS

Exercise 4.3

KEY CONCEPTS - random phenomena

This is a good opportunity to observe chance variation. When tossing a thumbtack 100 times, although each student will have the point face up on different tosses, the proportion of times in which the point faces up will not differ that much from student to student. In Chapter 5 you will learn how to calculate how much this proportion will vary from student to student. However, for now simply record the number of times the point faces up in the space below and use this to approximate the probability of the thumbtack landing point up. Be sure to use a hard surface as the probability of landing point up will be different if you toss the thumbtack on a carpet.

Number of times landing point up =

Approximate probability =

Exercise 4.7

KEY CONCEPTS - simulating a random phenomenon

a) You will need to use your computer software to simulate the 100 trials or the Applet. After simulating the 100 trials calculate the proportion of "hits."

proportion of hits =

For most students, their proportion of hits will be within 0.05 or 0.10 of the true probability of .5.

b) You need to go through your sequence to determine the longest string of hits or misses.

Longest run of shots hit = Longest run of shots missed =

COMPLETE SOLUTIONS

Exercise 4.3

Our 100 tosses yielded the following results. Your answers will not agree exactly, but may disagree due to the brand of thumbtack you are using as well as chance variation The length of the point and the size of the head of the thumbtack will vary from brand to brand, and these factors will affect the probability of landing point up. These difficulties do not arise when tossing pennies, as different pennies are much more similar to each other.

Number of times landing point up = 55

Approximate probability = 55/100 = 0.55 = 55%

Exercise 4.7

a) Our sequence of hits (H) and misses (M) is given below.

H	H	M	H	H	H	M	M	H	H	H	M	H	M	H
M	H	M	M	H	M	M	H	H	H	M	M	H	H	H
M	M	M	M	H	M	M	H	H	H	H	M	H	H	H
M	M	M	M	M	H	M	H	H	M	H	M	H	M	M
H	H	H	H	M	H	M	M	M	M	H	H	M	H	H
M	M	H	H	H	M	M	H	H	M	M	H	M	H	M
M	H	M	H	H	M	H	H	H	H					

proportion of hits = .54

b) You need to go through your sequence to determine the longest string of hits or misses. In our example,

Longest run of shots hit = 4 (this occurred more than once)
Longest run of shots missed = 5

SECTION 4.2

OVERVIEW

The description of a random phenomenon begins with the **sample space** which is the list of all possible outcomes. A set of outcomes is called an **event.** Once we have determined the sample space, a **probability model** tells us how to assign probabilities to the various events that can occur. There are four basic rules that probabilities must satisfy.

- Any probability is a number between 0 and 1.
- All possible outcomes together must have probability 1.
- The probability that an event does not occur is 1 minus the probability that the event occurs.
- If two events have no outcomes in common, the probability that one or the other occurs is the sum of their individual probabilities.

In a sample space with a finite number of outcomes, probabilities are assigned to the individual outcomes and the probability of any event is the sum of the probabilities of the outcomes that it contains. In some special cases, the outcomes are all **equally likely** and the probability of any event A is just computed as

$$P(A) = (\text{number of outcomes in } A)/(\text{number of outcomes in } S).$$

Events are **disjoint** if they have no outcomes in common. In this special case the probability that one or the other event occurs is the sum of their individual probabilities. This is the addition rule for disjoint events, namely

$$P(A \text{ or } B) = P(A) + P(B).$$

Events are **independent** if knowledge that one event has occurred does not alter the probability that the second event occurs. The mathematical definition of independence leads to the **multiplication rule** for independent events. If A and B are independent, then

$$P(A \text{ and } B) = P(A)P(B).$$

In any particular problem we can use this definition to check if two events are independent by seeing if the probabilities multiply according to the definition. However, most of the time, independence is assumed as part of the probability model. The four basic rules, plus the multiplication rule, allow us to compute the probabilities of events in many random phenomena.

Many students confuse independent and disjoint events once they have seen both definitions. Remember, disjoint events have no outcomes in common and when two events are disjoint, you can compute $P(A \text{ or } B) = P(A) + P(B)$ in this special case. The probability being computed is that one or the other event occurs. Disjoint events cannot be independent since once we know that A has occurred, then the probability of B occurring becomes 0 (B cannot have occurred as well - this is the meaning of disjoint). The multiplication rule can be used to compute the probability that two events occur simultaneously, $P(A \text{ and } B) = P(A)P(B)$, in the special case of independence.

GUIDED SOLUTIONS

Exercise 4.11

KEY CONCEPTS - sample space

One of the main difficulties encountered when describing the sample space is finding some notation to express your ideas formally. Following the text, our general format is $S = \{ \quad \}$, where a description of the outcomes in the sample space is included within the braces.

a) You want to express that any number between 0 and 24 is a possible outcome. So you would write S = {all numbers between 0 and 24}.

b) You may not want to put an upper bound on the amount, so you should allow for any number greater than 0, remembering that certain numbers are not possible values for the amount of change a student is carrying.

S =

c) $S =$

d) $S =$

Exercise 4.13

KEY CONCEPTS - applying the probability rules

a) Since these are the only blood types, what has to be true about the sum of the probabilities for the different types? Use this to find *P(*AB).

b) What's true about the events O and B blood type? Which probability rule do we follow? (Don't be confused by the wording in the problem which says "people with blood types O and B." In the context of this problem and the language of probability we are using, it really means O or B. There are no people with blood types O and B).

Exercise 4.21

KEY CONCEPTS - independent events

Although independence is often assumed in setting up a probability model, in other cases we must use the formal definition to determine if two events are independent. In order to determine if the events A = Hispanic and B = White are independent, we must see if they satisfy the multiplication rule. This requires that we evaluate $P(A)$, $P(B)$ and $P(A \text{ and } B)$. To evaluate $P(A)$ we need to add up the proportions of the population corresponding to each race that are also Hispanic.

$$P(A) = 0.000 + 0.003 + 0.060 + 0.062 = 0.125.$$

Now evaluate $P(B)$ and $P(A \text{ and } B)$ on your own, and determine if the multiplication rule is satisfied for these events.

$P(B) =$
$P(A \text{ and } B) =$
$P(A)P(B) =$

Exercise 4.29

KEY CONCEPTS - sample spaces for simple random sampling, probabilities of events

a) It's easy to make the list. *S* = {(Abby, Mei-Ling), (Abby, Julie), etc.}. There is no need to include both (Abby, Mei-Ling) and (Mei-Ling, Abby) in your list, since both refer to the same two individuals.
b) How many outcomes are there in the sample space in (a)? If they are equally likely, what is the probability of each?

c) How many outcomes in *S* include Mei-Ling? When the outcomes are equally likely, the probability of the event is just the

(number of outcomes in the event) / (number of outcomes in *S*).

d) How many outcomes in *S* include neither of the two men?

Exercise 4.31

KEY CONCEPTS: independence, multiplication rule

The probability of winning the major battle is 0.6. What is the probability of winning all three small battles? How would you decide which strategy is best? Write the event "winning all three small battles" in terms of winning each of the small battles and use the independence of victories or defeats in the small battles.

*P(*winning all three small battles) =

Which strategy do you prefer and why?

Exercise 4.35

KEY CONCEPTS - multiplication rule for independent events

a) The three years are independent. If *U* indicates a year for the price being up and *D* indicates a year for the price being down, you need to compute *P(UUU*).

b) Since the events are independent, what happens in the first two years does not affect the probability of going up or down in the third year. What's the probability of the price going down in any given year?

c) This problem must be set up carefully and done in steps.

Step 1 - Write the event of interest in terms of simpler outcomes. How would you write *P(UU* or *DD*) in terms of *P(UU*) and *P(DD*)?

*P(*moves in the same direction in the next two years) = *P(UU* or *DD*).

Step 2 - Evaluate *P(UU)* and *P(DD)* and substitute your answer in the expression from Step 1.

COMPLETE SOLUTIONS

Exercise 4.11

a) *S* = {all numbers between 0 and 24}.
b) *S* = {0, 0.01, 0.02, 0.03, ...}, where 0.01 corresponds to 1 cent, etc.
c) *S* = {A, B, C, D, F}. You may need to include +'s and -'s depending on the grading scheme for your school.
d) *S* = {no, yes}.

Exercise 4.13

a) The probabilities for the different blood types must add to 1. The sum of the probabilities for blood types O, A, and B is 0.45 + 0.40 + 0.11 = 0.96. Subtracting this from 1 tells us that the probability of the remaining type AB must be 0.04.

b) Maria can receive transfusions from people with blood types O or B. Since a person cannot have both of these blood types, they are disjoint. The calculation follows probability rule 4 which says *P*(O or B) = *P(*O) + *P(*B) = 0.45 + 0.11 = 0.56.

Exercise 4.21

$$P(B) = 0.060 + 0.691 = 0.751.$$

The probability of being both white and Hispanic corresponds to the single entry in the column labeled "Hispanic" and the row labeled "white." Reading the entry in the table gives *P(A* and *B*) = 0.060. Since *P(A)P(B)* = 0.125 × 0.751 = 0.094, the multiplication rule is not satisfied and the events are not independent.

Exercise 4.29

a) *S* = {(Abby, Mei-Ling), (Abby, Julie), (Abby, Sam), (Abby, Roberto), (Mei-Ling, Julie), (Mei-Ling, Sam), (Mei-Ling, Roberto), (Julie, Sam), (Julie, Roberto), (Sam, Roberto)}.

b) There are 10 possible outcomes. Since they are equally likely, each has probability 0.10.

c) Mei-Ling is in 4 of the outcomes so her chance of attending the conference in Paris is 4 / 10 = 0.4. (Note that each person has the same probability of going. Remember from Chapter 2 this is a property of a SRS).

d) The chosen group must contain two women, (Abby, Mei-Ling), (Abby, Julie), or (Mei-Ling, Julie). There are three possibilities, so the desired probability is 3 / 10 = 0.3.

Exercise 4.31

Denote the event that the general wins the first small battle by W_1, the event that the general wins the second small battle by using W_2, and the event that the general wins the third small battle by using W_3. Then

P(winning all three small battles) = $P(W_1$ and W_2 and $W_3)$ = $P(W_1)P(W_2)P(W_3)$

since victories or defeats in the small battles are independent. We know that the probability of winning each small battle is 0.8, so

P(winning all three small battles) = $P(W_1)P(W_2)P(W_3)$ = $0.8 \infty 0.8 \infty 0.8$ = 0.512.

Because the general is more likely to win the major battle than to win all three small battles, his strategy should be to fight one major battle.

Exercise 4.35

a) $P(UUU) = (0.65)^3 = 0.2746$

b) The probability of the price being down in any given year is 1 - 0.65 =0.35. Since the years are independent, the probability of the price being down in the third year is 0.35, regardless of what has happened in the first two years.

c) *P(*moves in the same direction in the next two years) = *P(UU* or *DD*) = *P(UU*) + *P(DD*), since the events *UU* and *DD* are disjoint. Using the independence of two successive years, $P(UU) = (0.65)^2 = 0.4225$, and $P(DD) = (0.35)^2 = 0.1225$. Putting this together,

*P(*moves in the same direction in the next two years) = 0.4225 + 0.1225 = 0.5450.

SECTION 4.3

OVERVIEW

A **random variable** is a variable whose value is a numerical outcome of a random phenomenon. The restriction to numerical outcomes makes the description of the probability model simpler and allows us to begin to look at some further properties of probability models in a unified way. If we toss a coin three times and record the sequence of heads and tails, then an example of an outcome would be HTH, which would not correspond directly to a random variable. On the other hand, if we were only keeping track of the number of heads on the three tosses, then the outcome of the experiment would be 0, 1, 2, or 3 and would correspond to the value of the random variable X = number of heads.

The two types of random variables we will encounter are **discrete** and **continuous** random variables. The **probability distribution** of a random variable tells us about the possible values of X and how to assign probabilities to these values. A discrete random variable has a finite number of values, and the probability distribution is a list of the possible values of X and the probabilities assigned to these values. The probability distribution can be given in a table or using a **probability histogram**. For any event described in terms of X, the probability of the event is just the sum of the probabilities of the values of X included in the event.

A continuous random variable takes all values in some interval of numbers. Probabilities of events are determined using a **density curve**. The probability of any event is the area under the curve corresponding to the values that make up the event. For density curves that involve regular shapes such as rectangles or triangles, we can compute probabilities of events using simple geometrical arguments. The **normal distribution** is another example of a continuous probability distribution, and probabilities of events for normal random variables are computed by standardizing and referring to Table A as was done in Section 1.3.

GUIDED SOLUTIONS

Exercise 4.45

KEY CONCEPTS - discrete random variables, computing probabilities

a) Write the event in terms of a probability about the random variable X. While you can figure out the answer without doing this, it's good practice to start using the notation for random variables. To find the probability that "the unit has 5 or more rooms," add the appropriate probabilities given in Exercise 4.43 that provides the distribution of X. (Be sure to use the row for owned units).

b) Express the event in words and compute its probability using the distribution of X in Exercise 4.43. How is this event different than the event in part (a).

c) You should have different answers in part (a) and part (b). What fact about discrete random variables does this illustrate? If X had a continuous distribution, would the events in (a) and (b) have the same probability?

Exercise 4.50

KEY CONCEPTS - finding the probability distribution of a random variable

a) The probability of a randomly selected student opposing the funding of interest groups is 0.4 and the probability of favoring it is 0.6. The opinions of different students sampled are independent of each other. So you can use the multiplication rule to find

*P(*A supports, B supports, and C opposes) =

b) It is easiest to do this by making a table to keep track of the calculations. The first entry is given below. There should be eight lines to the table when you're done. If you've done the calculations correctly, what should be true about the eight probabilities? Don't worry about the column labeled value of X for now. It will not be needed until part (c).

A	B	C	Probability	Value of X
support	support	support	$(0.6)^3 = 0.216$	

c) For each committee listed in the table in (b), find the associated value of X. For the first row, 0 people oppose the funding of interest groups so the value of X is 0 for a committee with these views. The values of X which can occur are 0, 1, 2, and 3. To find the probability that X takes any of these values, just add up the probabilities of the committees with that value of X. Fill in the table below with your values and make sure that the probabilities sum to 1.

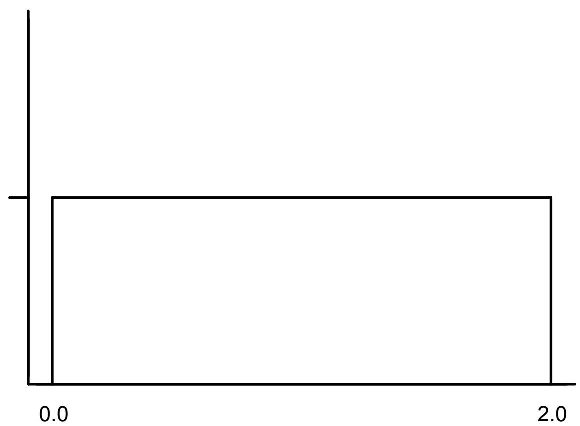
Value of X	Probability

d) If a majority oppose funding, how many people on the committee would have to oppose funding? What does this say about *X*? Now use the table you constructed in (c) to evaluate this probability.

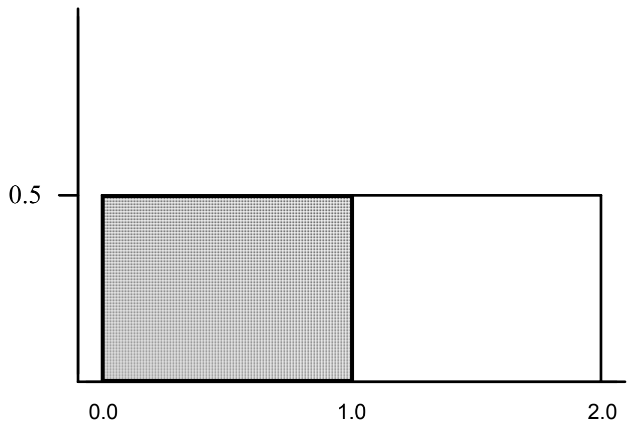
Exercise 4.53

KEY CONCEPTS - continuous random variables, computing probabilities

a) The graph of the density curve is given below. The property you are given is that the density has a constant height between 0 and 2. Since the area under the density curve must be equal to 1, what is the height? Recall that the area of a rectangle is the length $\times$ height.



b) As with finding areas under normal curves, it helps to draw a sketch of the density which includes the area corresponding the probability that you need to evaluate. In this part you need to find $P(Y \leq 1)$, when Y is a random number between 0 and 2. The density curve with the area corresponding to this probability is given below. Since it is a rectangular region, the area corresponds to the length $\times$ height = 0.5 $\times$ 1 = 0.5, which is the $P(Y \leq 1)$. Remember for continuous densities, $P(Y \leq 1) = P(Y < 1)$.


0.5
0.0
1.0
2.0

c) Sketch the density and the area you need below.

d) Sketch the density and the area you need below.

Exercise 4.55

KEY CONCEPTS - probabilities for a sample proportion

a) Finding probabilities associated with a sample proportion when we know the mean and standard deviation of the sampling distribution, requires first standardizing to a *z*-score so that we can refer to the table for the standard normal distribution. In this example the mean is $\mu = 0.30$, (the value of p), and the standard deviation is given as $\sigma = 0.023$. If you are still uncomfortable doing this type of problem, it is best to continue to draw a picture of a normal curve and the required area as we did in Section 1.3. Otherwise you can just follow the method of Example 4.18 in this chapter. At least half the sample corresponds to the sample proportion being greater than or equal to 0.5.

$P(\hat{p} \geq 0.5) =$

b) $P(\hat{p} < 0.25) =$

c) $P(0.25 < \hat{p} < 0.35) =$

COMPLETE SOLUTIONS

Exercise 4.45

a) $P(X \geq 5) = 0.210 + 0.224 + 0.197 + 0.149 + 0.053 + 0.035 = 0.868$

b) The event $\{X > 5\}$ indicates that "the unit has more than 5 rooms." The probability is computed as

$$P(X > 5) = 0.224 + 0.197 + 0.149 + 0.053 + 0.035 = 0.658$$

c) The answers are different because for discrete distributions the probability of an individual outcome is not necessarily zero. In this case, the probability of the individual outcome 5 is not zero, so that $P(X > 5)$ does not have the same value as $P(X \geq 5)$. (Note: For this distribution of X, $P(X > 4.5) = P(X \geq 4.5)$ since the outcome 4.5 has zero probability).

Exercise 4.50

a) $P($A supports, B supports and C opposes)
$= P($A supports) $P($B supports) $P($C opposes) $= (0.6)(0.6)(0.4) = 0.144$

b)

A	B	C	Probability	Value of X
support	support	support	$(0.6)^3 = 0.216$	0
support	support	oppose	$(0.6)^2(0.4) = 0.144$	1
support	oppose	support	$(0.6)^2(0.4) = 0.144$	1
oppose	support	support	$(0.6)^2(0.4) = 0.144$	1
support	oppose	oppose	$(0.6)(0.4)^2 = 0.096$	2
oppose	support	oppose	$(0.6)(0.4)^2 = 0.096$	2
oppose	oppose	support	$(0.6)(0.4)^2 = 0.096$	2
oppose	oppose	oppose	$(0.4)^3 = 0.064$	3
			Total = 1.00	

c) The value of X is given in the table in (b). The possible values of X are 0, 1, 2, and 3. To find the probability that X takes any value, just add up the probabilities of the committees with that value of X. For example, $P(X = 2) = 3(0.096) = 0.288$.

Value of X	Probability
0	0.216
1	0.432
2	0.288
3	0.064

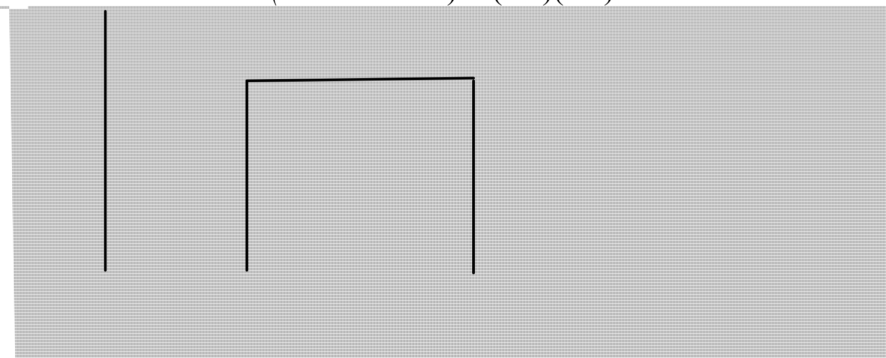
d) A majority oppose say that either 2 or 3 members of the board oppose. If 2 oppose then $X = 2$ and if 3 oppose then $X = 3$. So the event is $X \geq 2$ and the required probability is $P(X \geq 2) = 0.288 + 0.064 = 0.352$.

Exercise 4.53

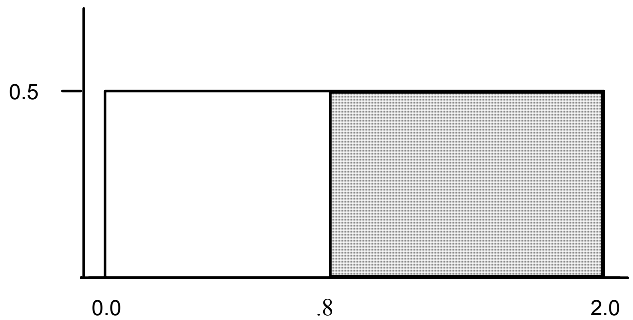
a) The area is the length × height. Since the area is 1 and we know the length is 2, you must solve the equation 2 × height = 1. Thus the height must be equal to 1/2 or 0.5.

b) See the Guided Solutions.

c) The shaded area is $P(0.5 < Y < 1.3) = (0.8)(0.5) = 0.40$.



d) The shaded area is $P(Y \geq 0.8) = (1.2)(0.5) = 0.60$.



Exercise 4.55

a) $$P(\hat{p} \geq 0.50) = P\left(\frac{\hat{p} - 0.30}{0.023} \geq \frac{0.50 - 0.30}{0.023}\right) = P(Z \geq 8.70) \approx 0$$

b) $$P(\hat{p} < 0.25) = P\left(\frac{\hat{p} - 0.30}{0.023} \leq \frac{0.25 - 0.30}{0.023}\right) = P(Z \leq -2.17) = 0.015$$

c) $$P(0.25 < \hat{p} < 0.35) = P\left(\frac{0.25 - 0.30}{0.023} \leq \frac{\hat{p} - 0.30}{0.023} \leq \frac{0.35 - 0.30}{0.023}\right)$$
$$= P(-2.17 \leq Z \leq 2.17) = 0.985 - 0.015 = 0.970.$$

SECTION 4.4

OVERVIEW

In Chapter 1 we introduced the concept of the distribution of a set of numbers or data. The distribution describes the different values in the set and the frequency or relative frequency with which those values occur. The mean of the numbers is a measure of the center of the distribution and the standard deviation is a measure of the variability or spread. These concepts are also used to describe features of a random variable X. The probability distribution of a random variable indicates the possible values of the random variable and the probability (relative frequency in repeated observations) with which they occur. The **mean** μ_X of a random variable X describes the center or balance point of the probability distribution or density curve of X. If X is a discrete random variable having possible values $x_1, x_2, \ldots, x_k$ with corresponding probabilities $p_1, p_2, \ldots, p_k$, the mean μ_X is the average of the possible values weighted by the corresponding probabilities, i.e.

$$\mu_X = x_1p_1 + x_2p_2 + \ldots + x_kp_k$$

The mean of a continuous random variable is computed from the density curve but computations require more advanced mathematics. The law of large numbers relates the mean of a set of data to the mean of a random variable and says that the average of the values of X observed in many trials approaches μ_X.

The **variance** σ_X^2 of a random variable X is the average squared deviation of the values of X from their mean. For a discrete random variable

$$\sigma_X^2 = (x_1 - \mu_X)^2p_1 + (x_2 - \mu_X)^2p_2 + \ldots + (x_k - \mu_X)^2p_k.$$

The **standard deviation** σ_X is the positive square root of the variance. The standard deviation measures the variability of the distribution of the random variable X about its mean. The variance of a continuous random variable, like the mean, is computed from the density curve. Again, computations require more advanced mathematics.

The mean and variances of random variables obey the following rules. If a and b are fixed numbers then

$$\mu_{a+bX} = a + b\mu_X$$

$$\sigma^2_{a+bX} = b^2\sigma^2_X$$

If X and Y are any two random variables, then

$$\mu_{X+Y} = \mu_X + \mu_Y$$

If X and Y are independent random variables, then

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y$$

$$\sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y$$

If X and Y have correlation ρ, then the general addition rule for variances of random variables is

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y + 2\rho\sigma_X\sigma_Y$$

$$\sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y - 2\rho\sigma_X\sigma_Y$$

GUIDED SOLUTIONS

Exercise 4.66

KEY CONCEPTS - independence, misconceptions about the law of large numbers

The key concept that must be properly understood to answer the questions raised in this problem is the notion of independence. Events *A* and *B* are independent if knowledge that *A* has occurred does not alter our assessment of the probability that *B* will occur. Do not be mislead by misconceptions based on a faulty understanding of the nature of random behavior or a faulty understanding of the law of large numbers (either that runs indicate that a hot streak is in progress and will continue for a while, or that a run of one type of outcome must be immediately balanced by a lack of the outcome for several trials).

Exercise 4.67

KEY CONCEPTS - rules for means and variances

a) Recall that if X and Y are any two random variables, then

$$\mu_{X+Y} = \mu_X + \mu_Y$$

In this problem, let X be the time to bring a part from a bin to its position on an automobile chassis and Y be the time required to attach the part to the chassis. Then the time for the total operation is $X + Y$. Use the information in the problem and the rule for the mean of $X + Y$ to evaluate the μ_{X+Y}, the mean time to complete the operation.

b) Do the variances affect the rules for means?

c) Does the correlation affect the rules for means?

Exercise 4.69

KEY CONCEPTS - rules for means and variances

Unlike the mean of $X + Y$, the variance of $X + Y$ depends on the correlation. Specifically the rule for variances is

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y + 2\rho\sigma_X\sigma_Y,$$

which simplifies to $\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y$ when $\rho = 0$. Recall that when X and Y are independent, $\rho = 0$. Thus, you can use the simpler formula when X and Y are independent, while you need to use the general rule when $\rho = 0.3$.
Again, let X be the time to bring a part from a bin to its position on an automobile chassis and Y be the time required to attach the part to the chassis. Then the time for the total operation is $X + Y$. The problem asks you to find the standard deviation of the total time, so after finding the variance you must take the square root of your answer. You are given $\sigma_X = 2$ and $\sigma_Y = 4$.

i) $\rho = 0$ $\sigma^2_{X+Y} =$ $\sigma_{X+Y} =$

ii) $\rho = .3$ $\sigma^2_{X+Y} =$ $\sigma_{X+Y} =$

If you did the calculations correctly, the variance should be larger when the times are positively correlated. Try and explain in simple language why that should be the case.

Exercise 4.74

KEY CONCEPTS - rules for means and variances of random variables

a) The distribution of X is

Temperature (x)	540	545	550	555	560
Probability (p)	0.1	0.25	0.3	0.25	0.1

Recall that the average (mean) of X = earnings, is computed using the formula

$$\mu_X = x_1p_1 + x_2p_2 + \ldots + x_kp_k$$

where the values of x_i and the p_i are given in the preceding table.

$\mu_X =$

Once you have calculated the mean μ_X, the general formula for the variance is

$$\sigma_X^2 = (x_1 - \mu_X)^2 p_1 + (x_2 - \mu_X)^2 p_2 + \ldots + (x_k - \mu_X)^2 p_k$$

After you have calculated the variance, remember to take the square root to obtain the standard deviation. If you are having trouble with these formulas, review Example 4.20 in the text.

$\sigma_X^2 =$

$\sigma_X =$

b) Calculations are relatively easy using the results of (a) if we recall the formulas

$$\mu_{a+bX} = a + b\mu_X$$

$$\sigma^2_{a+bX} = b^2\sigma^2_X$$

What are *a* and *b* in this case?

c) This is just like (b). Again one must identify *a* and *b*.

Exercise 4.79

KEY CONCEPTS - mean of a random variable

Recall that the average (mean) of X = earnings, is computed using the formula

$$\mu_X = x_1p_1 + x_2p_2 + \ldots + x_kp_k$$

where the values of x_i and the p_i are given in the following table. We have filled in the missing probability. How was it calculated?

Earnings (x)	-\$99750	-\$99500	-\$99250	-\$99000	-\$98750	\$1250
Probability (p)	0.00183	0.00186	0.00189	0.00191	0.00193	0.99058

Now use the formula to compute μ_X. The first five values of X are negative and the last is positive, so be careful with the signs of the products when computing the mean of X.

$\mu_X =$

COMPLETE SOLUTIONS

Exercise 4.66

a) Consecutive spins of a (fair) roulette wheel should be independent. Thus the particular results of previous spins will not change the probability of any particular outcome on the next spin. On the next spin black is just as likely as red. The gambler's reasoning that red is "hot" fails to recognize that spins are independent. It is based on a false understanding of random behavior.

b) The gambler is again wrong because he is assuming that consecutive cards are independent. This is not the case here. Initially, the deck contains 52 cards, half of which are red and half of which are black. However, each time a card is removed from the deck, the number of red and black cards remaining changes. For example, if I am dealt five red cards from a deck of 52 cards, the deck now contains only 47 cards of which 21 are red and 26 are black. The probability that the next card is red is 21/47 which is less than the probability that the next card is black, which is 26/47.

Exercise 4.67

a) We are told that $\mu_X = 11$ seconds and $\mu_Y = 20$ seconds. The mean time to complete the operation is

$$\mu_{X+Y} = \mu_X + \mu_Y = 11 + 20 = 31 \text{ seconds}$$

b) Changing the standard deviations does not change the rule for finding the mean of $X + Y$, so the mean time to complete the operation is still 31 seconds even if the standard deviations of each part of the operation are decreased.

c) The correlation affects the variance of $X + Y$, not its mean. The mean time to complete the operation is still 31 seconds even if the times required for the two steps are correlated.

Exercise 4.69

i) $\rho = 0$ $\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y = (2)^2 + (4)^2 = 20$

$\sigma_{X+Y} = \sqrt{20} = 4.47$ seconds

ii) $\rho = .3$ $\sigma^2_{X+Y} = \sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y + 2\rho\sigma_X\sigma_Y = (2)^2 + (4)^2 + (.3)(2)(4)$

$= 22.4$

$\sigma_{X+Y} = \sqrt{22.4} = 4.73$ seconds

When the times are positively correlated, if it takes longer than average time to bring the part from the bin to the chassis, it will also tend to take longer than average to attach the part making for a very long total time. Similarly, shorter than average times for bringing the part from the bin will be associated with shorter than average times to attach the part. This tends to make the total time further from the overall mean of 31 seconds in both directions, increasing the variability. If the times were independent, than a longer time to bring the part from the bin might have a shorter than average time to attach the part, and this cancellation brings the total time closer to the mean than when there is a positive correlation.

Exercise 4.74

a) We compute

$\mu_X = 540(0.1) + 545(0.25) + 550(0.3) + 555(0.25) + 560(0.1)$

$= 54 + 136.25 + 165 + 138.75 + 56$

$= 550$

$\sigma^2_X = (540 - 550)^2(0.1) + (545 - 550)^2(0.25) + (550 - 550)^2(0.25) +$
$(555 - 550)^2(0.25) + (560 - 550)^2(0.1)$

$= 10 + 6.25 + 0 + 6.25 + 10$

$= 32.5$

$\sigma_X = \sqrt{32.5} = 5.7$

b) Here we are interested in X - 550 so a = -550 and b = 1. Thus we have

$\mu_{-550+X} = -550 + 1\mu_X = -550 + 550 = 0$

$\sigma^2_{-550+X} = 1^2\sigma^2_X = 32.5$

hence

$$\sigma_{-550+X} = \sqrt{32.5} = 5.7$$

c) Here we want (9/5)X + 32 so a = 32 and b = 9/5. Thus

$$\mu_{32 + (9/5)X} = 32 + (9/5)\mu_X = 32 + (9/5)550 = 32 + 990 = 1022$$

$$\sigma^2_{32 + (9/5)X} = (9/5)^2\sigma^2_X = (81/25)32.5 = 105.3$$

hence

$$\sigma_{32 + (9/5)X} = \sqrt{105.3} = 10.26.$$

Exercise 4.79

We calculate,

$$\mu_X = 0.00183(-99750) + 0.00186(-99500) + 0.00189(-99250) + 0.00191(-99000) + 0.00193(-98750) + 0.99058(1250)$$

$$= -182.5425 - 185.07 - 187.5825 - 189.09 - 190.5875 + 1238.225$$

$$= 303.3525.$$

Not surprisingly, the mean is positive so the insurance company expects to make a little over $303 per policy that it sells. Like any form of gambling, in the long run the insurance companies will make money despite an occassional large payout.

SECTION 4.5

OVERVIEW

This section discusses a number of basic concepts and rules that are used to calculate probabilities of complex events. The **complement** A^c of an event A contains all the outcomes in the sample space that are not in A. It is the "opposite" of A. The **union** of two events A and B contains all outcomes in A, in B, or in both. The union is sometimes referred to as the event A or B. The **intersection** of two events A and B contains all outcomes that are in both A and B simultaneously. The intersection is sometimes referred to as the event A and B. We say two events A and B are **disjoint** if they have no outcomes in common.

The **conditional probability** of an event *B* given an event *A* is denoted $P(B|A)$ and is defined by

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

when $P(A) > 0$. In practice it can often be determined directly from the information given in a problem. Two events A and B are **independent** if $P(B|A) = P(B)$.

Other general rules of elementary probability are

- Legitimate values: $0 \le P(A) \le 1$ for any event A

- Total probability: $P(S) = 1$, where S denotes the sample space.

- Complement rule: $P(A^c) = 1 - P(A)$

- Addition rule: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

- Multiplication rule: $P(A \text{ and } B) = P(A)P(B|A)$

- Bayes rule: $P(A|B) = \dfrac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$

  provided $0 < P(A), P(B) < 1$

- For disjoint events: $P(A \text{ and } B) = 0$ and so $P(A \text{ or } B) = P(A) + P(B)$

- For independent events: $P(A \text{ and } B) = P(A)P(B)$

In problems with several stages, it is helpful to draw a tree diagram to guide you in the use of the multiplication and addition rules.

GUIDED SOLUTIONS

Exercise 4.89

KEY CONCEPTS - the addition rule

This is an application of the addition rule $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

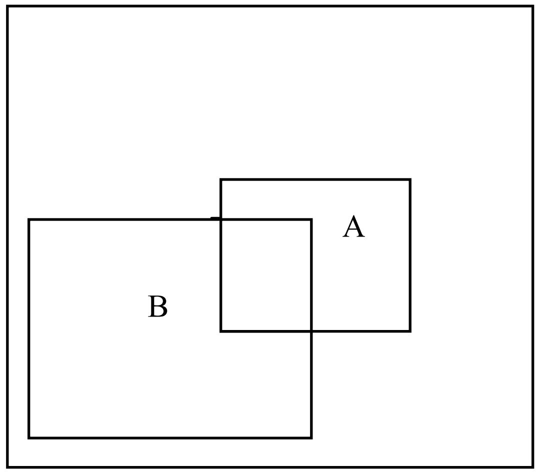
Exercise 4.91

KEY CONCEPTS - Venn diagrams

a) From Exercise 4.89 we have the following facts about *A* and *B*.

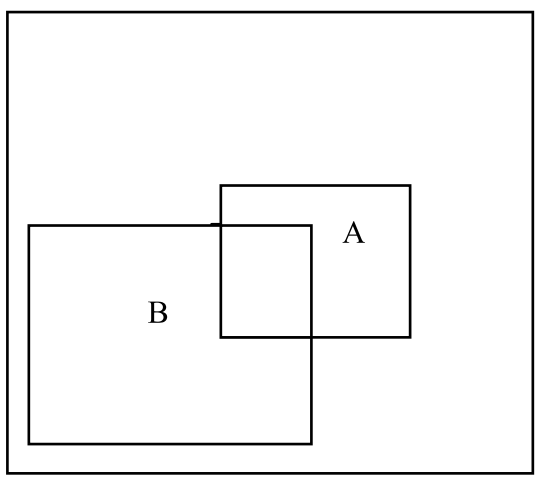
$P(A) = 0.6$
$P(B) = 0.5$
$P(A \text{ and } B) = 0.3$

Below is a Venn diagram showing *A* and *B*. The event that Consolidated wins both jobs corresponds to {*A* and *B*}. Shade the portion representing the event {*A* and *B*}in the Venn diagram below.



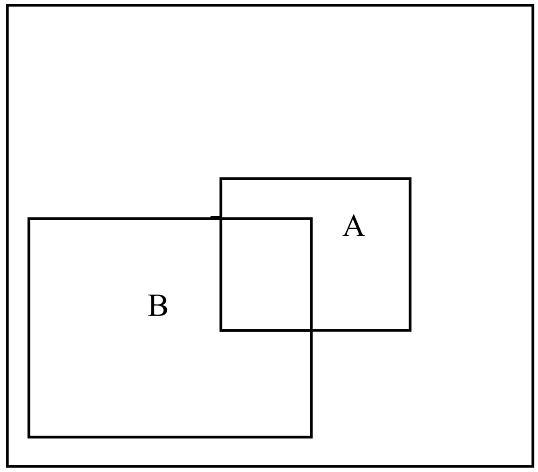
What is the probability that consolidated wins both jobs?

b) Below is a Venn diagram showing *A* and *B*. Shade the portion representing Consolidated wins the first job but not the second which is {*A* and B^c}.



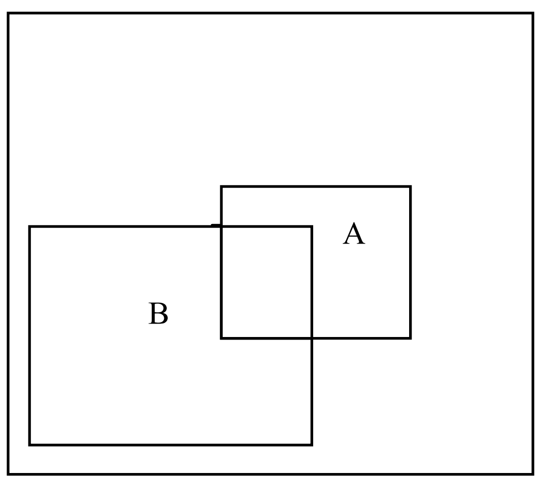
Use the facts listed in (a) and the Venn diagram to calculate *P(A* and B^c).

c) Below is a Venn diagram showing *A* and *B*. Shade the portion representing Consolidated does not win the first job but does the second. Write this event in terms of *A*, *B*, A^c and B^c.



Use the facts listed in (a) and the Venn diagram to calculate the probability of this event.

d) Below is a Venn diagram showing *A* and *B*. Shade the portion representing Consolidated does not win either job. Write this event in terms of *A*, *B*, A^c and B^c.



Use the facts listed in (a) and the Venn diagram to calculate the probability of this event.

Exercise 4.95

KEY CONCEPTS - conditional probabilities and the multiplication rule

The Table in Exercise 4.94 is reproduced below to assist you.

	Bachelor's	Master's	Professional	Doctorate	Total
Female	645 227	32	18 922		
Male	505	161 40	26	732	
Total	1150	388 72	44	1654	

a) You can calculate this probability directly from the table. All degree recipients in the table are equally likely to be selected (that is what it means to select a degree recipient at random) so that the fraction of the degree recipients in the table that are men is the desired probability. How many degree recipients are men? Where do you find this in the table? What is the total number of degree recipients represented in the table? Use these numbers to compute the desired fraction.

b) This probability can also be calculated directly from the table. Since this is a conditional probability (i.e. this is a probability given that the degree recipient is a man), we restrict ourselves only to degree recipients that are men. The desired probability is then the fraction of these men that received a bachelor's degree. Use the appropriate entries in the table to compute this fraction.

c) Recall the multiplication rule says

$P($degree recipient is both "male" and "received a bachelor's")

= *P(*degree recipient is male)*P(*degree recipient received bachelors | degree recipient is male)

Using the answers in (a) and (b), evaluate this probability.

The number of degree recipients that are both "male" and "received a bachelor's" can be read directly from the table. What is this number? What fraction of the total number of degree recipients represented in the table is this number? This should agree with the probability you calculated using the multiplication rule.

Exercise 4.101

KEY CONCEPTS - multiplication rules and conditional probability

We are given the following probabilities.

$P($call not completed$) = 0.70$

$P($talk to a man$) = 0.20$

$P($talk to a woman$) = 0.10$

$P($sale $\mid$ talk to a woman$) = 0.3$

$P($sale $\mid$ talk to a man$) = 0.2$

and

$P($sale $\mid$ call not completed$) = 0.0$

You are asked to compute the probability of a sale. Let A be the event "sale", B be the event "talk to a male," C the event "talk to a female," and D be the event "the call is not completed."

$$P(\text{sale}) = P(A) = P(A \text{ and } B) + P(A \text{ and } C) + P(A \text{ and } D)$$

First use the multiplication rule to evaluate $P(A \text{ and } B) = P(B)P(A \mid B)$, and similarly for $P(A \text{ and } C)$ and $P(A \text{ and } D)$, where we note that $P(A \text{ and } D) = 0$. Now combine your answers to evaluate

$P($sale$) =$

Exercise 4.103

KEY CONCEPTS - Bayes rule

What we want to calculate is

*P(*female | sale) = *P(*female and sale) / *P(*sale).

This is an application of Bayes rule as we are calculating the "reverse" conditional probability from the ones we are given. Although it is an example of Bayes rule, it is better to work your way through the problem as in Examples 4.35 and 4.36 of the text rather than trying to memorize the rule.

*P(*female and sale) can be calculated using the multiplication rule and *P(*sale) has been computed in Exercise 4.101. Put these together to give the desired probability.

COMPLETE SOLUTIONS

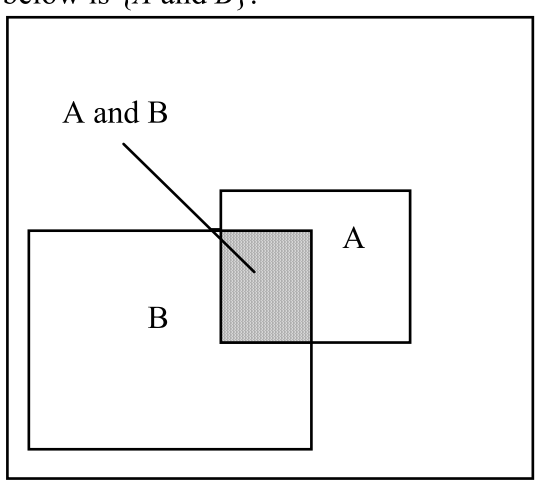
Exercise 4.89

We are given that $P(A) = 0.6$, $P(B) = 0.5$, and $P(A \text{ and } B) = 0.3$, hence

$$P(A \text{ or } B) = 0.6 + 0.5 - 0.3 = 0.8$$

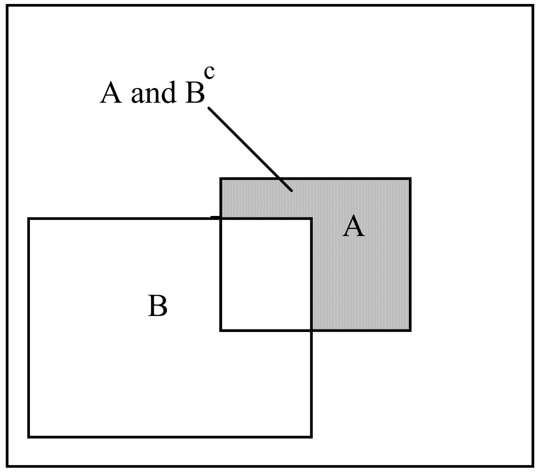
Exercise 4.91

a) The shaded area below is {*A* and *B*}.



$P(A$ and $B)$ is given as 0.3.

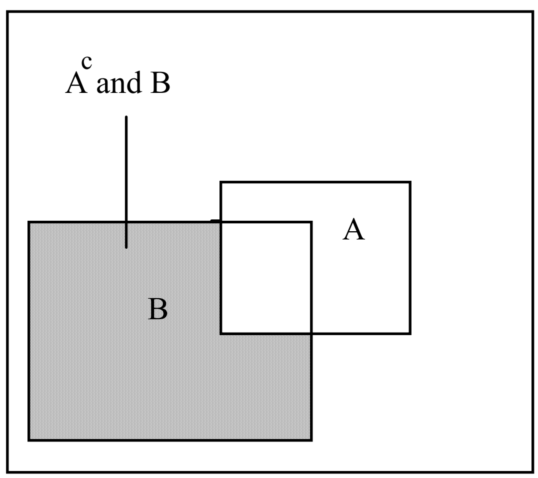
b)The shaded area below is {A and B^c}.



From the diagram we see that we can write

$$P(A \text{ and } B^c) = P(A) - P(A \text{ and } B) = 0.6 - 0.3 = 0.3$$

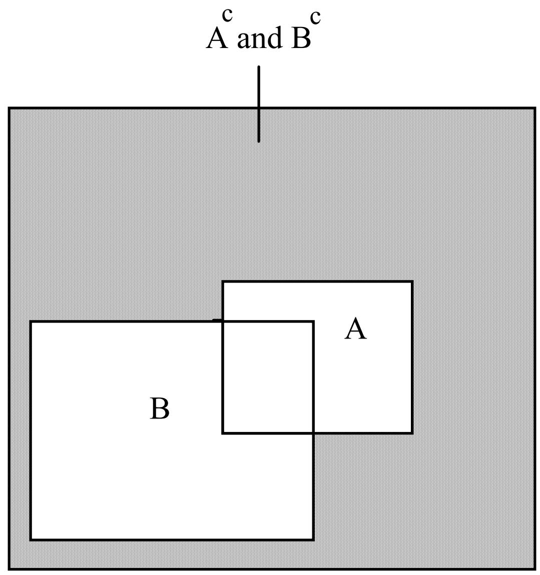
c) The shaded area below is { A^c and B }.



From the diagram we see that we can write

$$P(A^c \text{ and } B) = P(B) - P(A \text{ and } B) = 0.5 - 0.3 = 0.2$$

d) The shaded below area is { A^c and B^c }.



From the diagram we see that we can write

$$P(A^c \text{ and } B^c) = 1 - P(A \text{ or } B) = 1 - 0.8 = 0.2$$

where we have used the value of $P(A$ or $B) = 0.8$ calculated in problem 4.89.

Exercise 4.95

a) The number of degree recipients that are men is found in at the end of the row labeled "male" and is (in thousands) 732 The total number of degree recipients in the table is in the lower right corner and is (in thousands) 1654. The desired probability is thus

(number of degree recipients that are male)/(number of degree recipients)

= 732/1654

= 0.4426

b) The desired probability is

(number of males who received bachelors)/(number of males)

= 505/732

$$= 0.6899$$

c) Recall the multiplication rule says

*P(*degree recipient is both "male" and "received a bachelor's")

= *P(*degree recipient is male)*P(*degree recipient received bachelors | degree recipient is male)

= (0.4426)(0.6899) = 0.3053

The number of degree recipients that are both "male" and "received a bachelor's" can be read directly from the table and is (in thousands) 505 of the 1654 degree recipients, giving a probability of 505/1654 = 0.3053, which agrees with the answer obtained using the multiplication rule.

Exercise 4.101

KEY CONCEPTS - multiplication rules and conditional probability

$P(A \text{ and } B) = P(B)P(A \mid B) = (0.2)(0.2) = 0.04$

$P(A \text{ and C}) = P(\text{C})P(A \mid \text{C}) = (0.1)(0.3) = 0.03$

and

*P(*sale) = 0.04 + 0.03 = 0.07

Exercise 4.103

*P(*female | sale) = *P(*female and sale) / *P(*sale)

*P(*female and sale) = *P(*sale | female)*P(*female) = (0.3)(0.1) = 0.03

and

*P(*sale) = 0.07 from Exercise 1.01, giving

*P(*female | sale) = 0.03/0.07 = 0.4286